An Introduction to Finite Difference Methods

Gary Schurman, MBE, CFA

October, 2011

Finite difference methods (FDM) are a fundamental tool for the numerical pricing of derivative products in quantitative finance. Finite differences are used to approximate the values of derivatives of a given function f at a point t by values of f at a finite number of points near t. We are given the following infinitely differentiable function $f(t)$ where t equals time and θ is a constant equal to 0.80...

$$f(t)=e^{t\theta} \tag{1}$$

We are interested in the values of the function over the time interval [0,2]. The values of our function at the end points of the time interval are...

$$f(0)=e^{(0)(0.80)}=1.00 \;\; ...and... \;\; f(2)=e^{(2)(0.80)}=4.95 \tag{2}$$

Our task is to start at $t=0$ and use finite difference methods to approximate the value of our function at $t=2$. Given a Taylor Series Expansion of the second order where t equals the current time period and Δ equals the change in time an approximation for the value of our function at time $t+\Delta$ is...

$$f(t+\Delta)\approx f(t)+\frac{\delta f(t)}{\delta t}\Delta+\frac{1}{2}\frac{\delta^2 f(t)}{\delta t^2}\Delta^2 \tag{3}$$

We are given that our function evaluated at $t=0$ equals one. One possible approximation for the value of our function at $t=2$ would be to define Δ to be the total change in time and use Equation (3) as the approximating equation. To do this we need the first and second derivatives of our function with respect to time. The first and second derivatives of Equation (1) with respect to the variable t are...

$$f'(t)=\frac{\delta f(t)}{\delta t}=\theta e^{t\theta} \;\; ...and... \;\; f''(t)=\frac{\delta^2 f(t)}{\delta t^2}=\theta^2 e^{t\theta} \tag{4}$$

Using the equations above the approximation for $f(2)$ where $\Delta=2$ is...

$$\begin{aligned} f(2) &= f(0)+f'(0)\,(2-0)+\frac{1}{2}\,f''(0)\,(2-0)^2 \\ &= 1.00+1.60+1.28 \\ &= 3.88 \end{aligned} \tag{5}$$

Per Equation (2) the actual value of our function at $t=2$ is 4.95 whereas the approximating Equation (5) yields a value of 3.88. Remember that a Taylor Series Expansion is not very accurate for large values of Δ. Our plan will be to divide $\Delta=2$ into smaller time steps and use finite difference methods to obtain a more accurate approximation of our function's value at $t=2$.

A forward finite difference equation is represented by a positive Δ. Given that we want to find the value of $f(t+\Delta)$ the Taylor Series Expansion applicable to the function in Equation (1) is...

$$f(t+\Delta)=f(t)+f'(t)\Delta+\frac{1}{2}f''(t)\Delta^2+\frac{1}{6}f'''(t)\Delta^3+... \tag{6}$$

A backward finite difference equation is represented by a negative Δ. Given that we want to find the value of $f(t-\Delta)$, and noting that a negative Δ raised to a negative power is a negative number, the Taylor Series Expansion applicable to the function in Equation (1) is...

$$\begin{aligned} f(t+(-\Delta)) &= f(t)+f'(t)(-\Delta)+\frac{1}{2}f''(t)(-\Delta)^2+\frac{1}{6}f'''(t)(-\Delta)^3+... \\ f(t-\Delta) &= f(t)-f'(t)\Delta+\frac{1}{2}f''(t)\Delta^2-\frac{1}{6}f'''(t)\Delta^3+... \end{aligned} \tag{7}$$

FDM Approximations for the First Derivative

For the forward first order difference we will use Equation (6) and eliminate all items second order or greater. The rewritten equation is...

$$f(t+\Delta) = f(t) + f'(t)\Delta \tag{8}$$

Using Equation (8) and solving for $f'(t)$ the forward finite difference approximation for the first derivative is...

$$f'(t) = \frac{f(t+\Delta) - f(t)}{\Delta} \tag{9}$$

For the backward first order difference we will use Equation (7) and eliminate all items second order or greater. The rewritten equation is...

$$f(t-\Delta) = f(t) - f'(t)\Delta \tag{10}$$

Using Equation (10) and solving for $f'(t)$ the backward finite difference approximation for the first derivative is...

$$f'(t) = \frac{f(t) - f(t-\Delta)}{\Delta} \tag{11}$$

For the centered first order difference we will subtract Equation (10) from Equation (8) and solve for $f'(t)$. The centered finite difference approximation for the first derivative is...

$$\begin{aligned} f(t+\Delta) - f(t-\Delta) &= f(t) + f'(t)\Delta - f(t) + f'(t)\Delta \\ 2f'(t)\Delta &= f(t+\Delta) - f(t-\Delta) \\ f'(t) &= \frac{f(t+\Delta) - f(t-\Delta)}{2\Delta} \end{aligned} \tag{12}$$

FDM Approximations for the Second Derivative

For the second order difference we will use Equations (6) and (7) and eliminate all items third order or greater. The rewritten Equation (6) is...

$$f(t+\Delta) = f(t) + f'(t)\Delta + \frac{1}{2}f''(t)\Delta^2 \tag{13}$$

The rewritten Equation (7) is...

$$f(t-\Delta) = f(t) - f'(t)\Delta + \frac{1}{2}f''(t)\Delta^2 \tag{14}$$

For the second order difference we will add Equations (13) and (14) and solve for $f''(t)$. The second order approximation of the second derivative is therefore...

$$\begin{aligned} f(t+\Delta) + f(t-\Delta) &= 2f(t) + f''(t)\Delta^2 \\ f''(t) &= \frac{f(t+\Delta) - 2f(t) + f(t-\Delta)}{\Delta^2} \end{aligned} \tag{15}$$

The Problem Solution Using First Order Differences

Because we are given the value of $f(0)$ and want to find the future value of $f(2)$ we will be working with forward finite differences. If we take the forward finite difference approximation for the first derivative Equation (9) and solve for $f(t+\Delta)$ we get Equation (8), which is...

$$f(t+\Delta) = f(t) + f'(t)\Delta \tag{16}$$

If we define $\Delta = 0.20$ and start at $t = 0$ applying Equation (16) at every time step then the approximate value of our function at $t = 2$ is 4.65 per the table below...

t	$f(t)$	$f'(t)$	$f(t+\Delta)$	
0.00	1.00	0.80	1.16	
0.20	1.16	0.94	1.35	
0.40	1.35	1.10	1.57	
0.60	1.57	1.29	1.83	
0.80	1.83	1.52	2.13	
1.00	2.13	1.78	2.49	
1.20	2.49	2.09	2.90	
1.40	2.90	2.45	3.39	* See example calculation
1.60	3.39	2.88	3.97	
1.80	3.97	3.38	4.65	
2.00	4.65			

* Example calculation for $f(1.60)$:

$$\begin{aligned}
f(t+\Delta) &= f(t) + f'(t)\Delta \\
f(1.40+0.20) &= f(1.40) + f'(1.40)(0.20) \\
f(1.60) &= 2.90 + (0.80)e^{(1.40\times 0.80)}(0.20) \\
&= 2.90 + (2.45)(0.20) \\
&= 3.39
\end{aligned} \tag{17}$$

Whereas the actual value of $f(2)$ is 4.95 the approximation using first order differences gave us an approximate value of 4.65, which is much better than the approximation of 3.88 in Equation (5) above.

The Problem Solution Using Second Order Differences

If we take the finite difference approximation for the second derivative Equation (9) and solve for $f(t+\Delta)$ we get the following equation...

$$f(t+\Delta) = 2f(t) + f''(t)\Delta^2 - f(t-\Delta) \tag{18}$$

If we define $\Delta = 0.20$ and start at $t = 0$ applying Equation (18) at every time step then the approximate value of our function at $t = 2$ is 5.07 per the table below...

t	$f(t)$	$f(t-\Delta)$	$f''(t)$	$f(t+\Delta)$	
0.00	1.00	0.84	0.64	1.19	** See example calculation
0.20	1.19	1.00	0.75	1.40	
0.40	1.40	1.19	0.88	1.65	
0.60	1.65	1.40	1.03	1.94	
0.80	1.94	1.65	1.21	2.29	
1.00	2.29	1.94	1.42	2.68	
1.20	2.68	2.29	1.67	3.15	
1.40	3.15	2.68	1.96	3.69	*** See example calculation
1.60	3.69	3.15	2.30	4.33	
1.80	4.33	3.69	2.70	5.07	
2.00	5.07				

** Note that we were not given a value for $f(-0.20)$, which is $f(t-\Delta)$ in row one of our table above. We will approximate this value using the backward first order difference Equation (10). The approximation for $f(t-\Delta)$ in row one is...

$$\begin{aligned}
f(t-\Delta) &= f(t) - f'(t)\Delta \\
f(0-0.20) &= f(0) - f'(0)(0.20) \\
f(-0.20) &= 1.00 - (0.80)e^{(0\times 0.80)}(0.20) \\
&= 0.84
\end{aligned} \tag{19}$$

*** Example calculation for $f(1.60)$:

$$
\begin{aligned}
f(t+\Delta) &= 2f(t) + f''(t)\Delta - f(t-\Delta) \\
f(1.40+0.20) &= 2f(1.40) + f''(1.40)(0.20) \\
f(1.60) &= (2)(3.15) + (0.80)^2 e^{(1.40\times 0.80)}(0.04) - 2.68 \\
&= 6.30 + 0.0784 - 2.68 \\
&= 3.69
\end{aligned} \tag{20}
$$

Whereas the actual value of $f(2)$ is 4.95 the approximation using second order differences gave us an approximate value of 5.07, which is much better than the first order approximation of 4.65 in the table above and 3.88 in Equation (5) above.